

Eigenvalues of Rectangular Waveguide Using FEM With Hybrid Elements

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Abstract: A finite element analysis using hybrid triangular-rectangular elements is developed to estimate eigenvalues of a rectangular waveguide. Use of rectangular vector-edge finite elements in the vicinity of the PEC boundary and triangular elements in the interior region more accurately models the physical nature of the electromagnetic field, and consequently quicken the convergence.

I Introduction:

The Finite Element Method (FEM) in its pure as well as in its hybrid version form has been a very adaptive and powerful technique for close as well as open region electromagnetic (EM) simulations. Its geometrical and material adaptabilities are well suited to solve EM problems involving multilayer composites of arbitrary shapes. However, use of only triangular elements for 2-D case or use of only the tetrahedron elements for 3-D case results in a violation of boundary conditions. For example: according to the EM theory, the electric field close to a perfect conducting (PEC) plane must be perpendicular to the PEC plane. However, the inclined arms of triangular elements for 2-D case (or inclined arms of a tetrahedron element for 3-D case) intersecting the PEC plane make electric field non-normal to the PEC. So far, this problem has been mitigated by use of large number of basic triangular /tetrahedron elements so as to satisfy numerically above mentioned boundary condition. A similar problem is also encountered in analyzing thin composite layer in a large homogeneous medium. In the case of very thin composite layer, a large number of tetrahedron cells must be used to capture true nature of the EM field. Use of many sub cells inside a thin layer forces one to use large number basic cells over the rest of homogeneous volume of the problem and hence increases the total number of unknowns. In the present work, use of hybrid type of elements is proposed to address this problem.

In this paper, the eigenvalues of a rectangular waveguide using triangular, rectangular or triangular/rectangular elements with edge based basis function is solved. The convergence of these eigenvalues is studied as a function of number of triangular, rectangular or triangular/rectangular elements. When triangular/rectangular elements are used, the rectangular based elements are used in the region close to the walls of rectangular waveguide and triangular elements are used for the rest of the interior region. The rectangular elements for the region close to the metallic walls is selected because the arms of rectangular elements will always insure that in the close proximity of the PEC the electric fields are always perpendicular to the PEC plane. The convergence of numerical values of the eigenvalues of the rectangular waveguide as a function of number of hybrid elements studied and compared with the eigenvalues obtained using only triangular and rectangular elements. It has been proved that the eigenvalues converge to their final values at a faster rate when the hybrid elements are used.

II Theory:

(a) Triangular Elements: For a homogeneous rectangular waveguide, the transverse electric field \vec{E}_t for TE to z modes satisfies vector wave equation

$$\nabla \times (\nabla \times \vec{E}_t) + k_c^2 \vec{E}_t = 0 \quad (1)$$

Similar vector wave equation is valid for the transverse magnetic field \vec{H}_t for TM to z modes. In (1) k_c^2 is the cut off wave number. The wave equation in (1) can be posed into a weak form as

$$\int \int_{Cross} (\nabla \times \vec{T} \bullet \nabla \times \vec{E}_t - k_c^2 \vec{T} \bullet \vec{E}_t) dx dy = 0 \quad (2)$$

where \vec{T} is a testing function. To construct an approximate solution of (2) by the FEM, the rectangular cross section is approximated by an union of small triangles as shown in Figure 1(a), by an union of small rectangles or by an union of combination of triangles and rectangles as shown in Figure 1(b). On the triangles \vec{E}_t is approximated by a linear combination of functions $\vec{W}_i^t(x, y)$ [1,3] as

$$\vec{E}_t = \sum_{i=1}^3 \Gamma_i \vec{W}_i^t(x, y) \quad (3)$$

Substitution of (3) into (2) and selecting \vec{T} to be the same as $\vec{W}_i^e(x, y)$, the equation (2) can put into standard eigenvalue form as

$$[S][\Gamma] = k_c^2 [S^1][\Gamma] \quad (4)$$

where the elements of matrices are obtained

$$S_{IJ} = \int \int_e \nabla \times \vec{W}_J^t \bullet \nabla \times \vec{W}_I^t dx dy \quad \text{and} \quad S^1_{IJ} = \int \int_e \vec{W}_J^t \times \vec{W}_I^t dx dy$$

Equation (4) can be used to obtain the eigenvalues of rectangular waveguide as a function of number of triangles

(b) Triangular/Rectangular Elements: From Figure 1(a) it is clear that the transverse electric field close to the walls of the rectangular waveguide approach at an angle depending upon the arms of triangles whose nodes are on the walls. In order to satisfy normality conditions on the walls, large number of small triangles must be used in the proximity of the walls. This will certainly slow the convergence of eigenvalues as a function of number of triangular elements. To circumvent this problem the rectangular region is divided into two regions, region I and II, as shown in Figure 1(b). The region I is approximated by unions of small rectangular, where as the region II is approximated by a union of triangular elements as shown in Figure 1(b). Expression (3) is valid for expressing the transverse electric field in the region II, where as the transverse electric field in the region I is expressed as [2]

$$\vec{E}_t = \sum_{i=1}^4 \Gamma_i \vec{W}_i^r(x, y) \quad (5)$$

where $\vec{W}_i^r(x, y)$ is the vector basis function for the edge of a rectangular element in the region I. Using (3) and (5), the global matrices $[S]$ and $[S^1]$ can be obtained and the

eigenvalues of rectangular waveguide can be obtained as a function of number of triangles and rectangular elements.

III Numerical Results:

Three different meshing strategies were implemented for finite element analysis of an empty rectangular waveguide with $a/b = 2$ so the results could be compared for computational efficiency and rate of convergence. The first two schemes exclusively use only triangular or only rectangular vector-edge finite elements to fill the entire cross section to compute the eigenvalues of an empty rectangular waveguide. In the third scheme a single algorithm capable of identifying and handling finite elements of different types (rectangular or triangular) within the same cross section was implemented to compute eigenvalues of rectangular waveguide.

Computation of the eigenvalues of the empty rectangular waveguide was executed using only triangular, only rectangular, and hybrid meshes, and the results were compared. The number of edges contained within each mesh varied between roughly 300 –2100, and the relative accuracy of each strategy was plotted as a function of the number of edges for each, illustrating the convergence characteristics.

Figure 2 shows convergence characteristics for the first three modes of an empty rectangular waveguide with aspect ratio $a/b = 2$. Figure 2(a) shows convergence of the first eigenvalue. The plots of convergence in Figure 2(a) show that the hybrid approach converges quicker to the ideal eigenvalue (i.e. $\pi/2$) than either of other two meshes. Figures 2(b)-2(c) shows convergence for the second and third modes, respectively, with similar convergence results. Use of hybrid mesh yields quicker converges to the ideal eigenvalue (i.e. π) compared to the use of pure triangular or rectangular mesh. These results generalize to the majority of other modes studied.

IV Conclusion:

It is shown that the finite element modeling using the rectangular vector -edge finite elements on the PEC boundaries more accurately models the physical nature of the electromagnetic fields. As a consequence, the hybrid triangular –rectangular meshing strategy converges quicker and more precisely to the ideal eigenvalues compared to the only triangular element meshing or only rectangular meshing strategies.

V References

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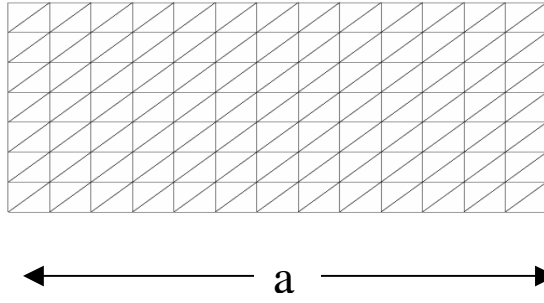


Figure 1(a) Rectangular waveguide discretized using triangular elements only

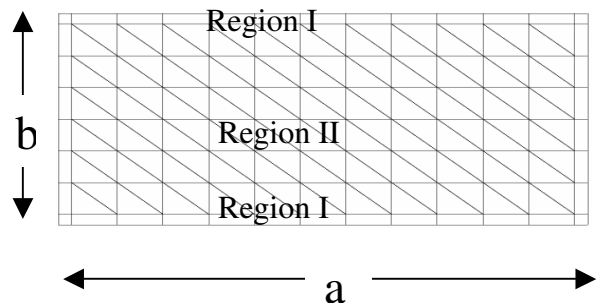


Figure 1 (b) Rectangular waveguide discretized using rectangular elements in Region I and triangular elements in interior Region II

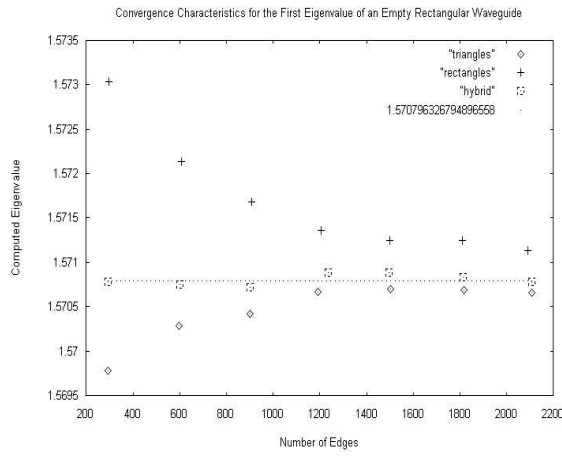


Figure 2(a)

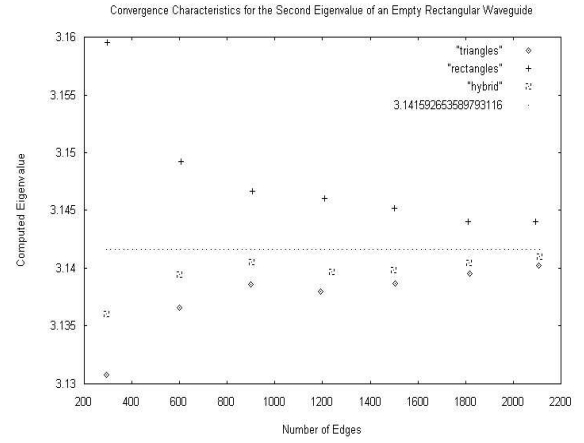


Figure 2(b)

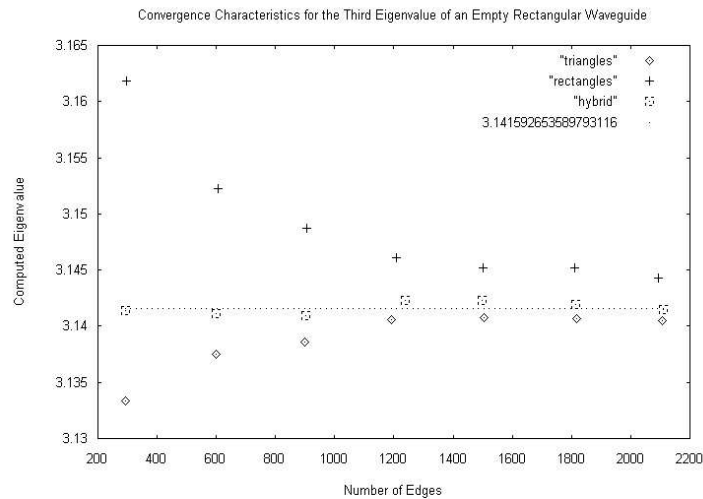


Figure 2©

Figure 2 Convergence characteristics for the first three modes of an empty rectangular waveguide with aspect ratio $a/h=2$.